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1983 J. Phys. A: Math. Gen. 16 997

(http://iopscience.iop.org/0305-4470/16/5/018)

J. Phys. A: Math. Gen. 16 (1983) 997–1004. Printed in Great Britain

Static, vacuum, cylindrical and plane symmetric solutions of the quadratic Poincaré gauge field equations

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Received 31 August 1982, in final form 20 October 1982

Abstract. We present some static, cylindrical and plane symmetric solutions to the equations of the quadratic Poincaré gauge field theory developed by Hehl and co-workers.

1. Introduction

This paper contains some preliminary results in a search for static, vacuum, cylindrical and plane symmetric solutions to the equations of the quadratic Poincaré gauge (QPG) field theory developed by Hehl and co-workers (see Hehl 1979, Baekler *et al* 1980). In § 2 a brief summary of the notation is given and the equations of the QPG theory are stated. For a concise description of the theory together with its physical motivation the reader is referred to Hehl *et al* (1980). A solution of the QPG field equations determines a Riemann–Cartan space–time which is specified by an orthonormal tetrad field (or, equivalently, a metric) and a metric-compatible non-symmetric connection. In the spherically symmetric solutions of Baekler *et al* (1980) and Baekler (1982) the metric has the property of satisfying Einstein's equations with the cosmological constant:

$$\tilde{R}_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{1.1}$$

where $\tilde{R}_{\alpha\beta}$ is the Ricci tensor for the symmetric Riemannian connection defined by the metric and the constant Λ involves certain coupling constants that occur in the QPG equations. Guided by these results we restrict ourselves here to looking for solutions which have this property. Accordingly, in § 3, the complete solution of (1.1) for static, cylindrical and plane symmetric metrics is derived and, in § 4, a number of special solutions to the QPG equations are derived corresponding to the metrics found in § 3.

2. The QPG vacuum equations

The underlying space–time is taken to be a differentiable manifold with normal hyperbolic metric g and connection ∇. It is assumed that the connection is compatible with the metric in the sense that

$$X\{g(Y,Z)\} = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{2.1}$$

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for arbitrary vector fields X, Y and Z.

Let e_α ($\alpha = 0, 1, 2, 3$) be an orthonormal tetrad field so that $g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. In terms of a local coordinate system $\{x^i\}$, $e_\alpha = e^i_{\ \alpha}(x)\partial_i$ where $\partial_i = \partial/\partial x^i$. The dual basis of one-forms will be denoted by $\theta^\alpha = e^\alpha_{\ i}(x)\,\mathrm{d}x^i$, where $e^\alpha_{\ i} e^i_{\ \beta} = \delta^\alpha_\beta$, and their exterior derivatives (the object of anholonomity in Hehl's terminology) may be written in the form

$$\mathrm{d}\theta^\alpha = \tfrac{1}{2}\Omega_{\mu\nu}{}^\alpha \theta^\mu \wedge \theta^\nu \tag{2.2}$$

where

$$\Omega_{\mu\nu}{}^\alpha = -\Omega_{\nu\mu}{}^\alpha = 2\partial_{[i} e^\alpha_{\ j]} e^i_{\ \mu} e^j_{\ \nu} \tag{2.3}$$

and the square brackets denote antisymmetrisation.

The connection one-forms $\omega_\alpha{}^\beta$ are defined by

$$\nabla_X e_\alpha = \omega_\alpha{}^\beta(X) e_\beta \tag{2.4}$$

for an arbitrary vector field X, so that

$$\omega_\alpha{}^\beta = \Gamma_{\mu\alpha}{}^\beta \theta^\mu \tag{2.5}$$

where

$$\nabla_{e_\mu} e_\alpha = \Gamma_{\mu\alpha}{}^\beta e_\beta. \tag{2.6}$$

Since $g(e_\alpha, e_\beta)$ are constants, it follows from (2.1) and (2.6) that

$$\Gamma_{\alpha\beta\gamma} = -\Gamma_{\alpha\gamma\beta} \tag{2.7}$$

and hence

$$\Gamma_{\alpha\beta\gamma} = \Gamma_{[\alpha\beta]\gamma} - \Gamma_{[\beta\gamma]\alpha} + \Gamma_{[\gamma\alpha]\beta}. \tag{2.8}$$

The torsion two-forms are given by

$$\Theta^\alpha = \mathrm{d}\theta^\alpha + \omega_\mu{}^\alpha \wedge \theta^\mu = \tfrac{1}{2}F_{\mu\nu}{}^\alpha \theta^\mu \wedge \theta^\nu \tag{2.9}$$

where, by (2.2) and (2.5),

$$F_{\mu\nu}{}^\alpha = \Omega_{\mu\nu}{}^\alpha + 2\Gamma_{[\mu\nu]}{}^\alpha \tag{2.10}$$

and hence, by (2.8),

$$\Gamma_{\mu\nu\alpha} = \tfrac{1}{2}(-\Omega_{\mu\nu\alpha} + \Omega_{\nu\alpha\mu} - \Omega_{\alpha\mu\nu} + F_{\mu\nu\alpha} - F_{\nu\alpha\mu} + F_{\alpha\mu\nu}). \tag{2.11}$$

The curvature two-forms are defined by

$$\Omega_\beta{}^\alpha = \mathrm{d}\omega_\beta{}^\alpha + \omega_\mu{}^\alpha \wedge \omega_\beta{}^\mu = \tfrac{1}{2}F_{\mu\nu\beta}{}^\alpha \theta^\mu \wedge \theta^\nu, \tag{2.12}$$

so that

$$F_{\mu\nu\beta\alpha} = \partial_\mu \Gamma_{\nu\beta\alpha} - \partial_\nu \Gamma_{\mu\beta\alpha} + \Gamma_{\mu\sigma\alpha}\Gamma_{\nu\beta}{}^\sigma - \Gamma_{\nu\sigma\alpha}\Gamma_{\mu\beta}{}^\sigma + \Gamma_{\sigma\beta\alpha}\Omega_{\mu\nu}{}^\sigma \tag{2.13}$$

where $\partial_\beta = e_\beta = e^i_{\ \beta}\partial_i$. Finally, for later use in the field equations, one defines the modified torsion components

$$T_{\alpha\beta\gamma} = F_{\alpha\beta\gamma} + 2\eta_{[\alpha|\gamma|}F_{\beta]}, \tag{2.14}$$

where

$$F_\beta = F_{\beta\gamma}{}^\gamma \tag{2.15}$$

$$T_\beta = T_{\beta\gamma}{}^\gamma \qquad \text{and} \qquad \Gamma_\beta = \Gamma_{\gamma\beta}{}^\gamma. \tag{2.16}$$

The vacuum field equations of the quadratic Poincaré gauge field theory for a particular choice of Lagrangian are given by equations (2.5) and (2.6) of Baekler *et al* (1980). These equations are written in terms of mixed coordinate and tetrad indices. Writing them entirely in terms of tetrad components one obtains the following equations (Hehl, private communication):

$$\begin{aligned}\Sigma_{\alpha\beta} \equiv{}& \partial_\gamma T^\gamma{}_{\alpha\beta} + (\Gamma_\gamma - \tfrac{1}{2}T_\gamma)T^\gamma{}_{\alpha\beta} - \Gamma_{\gamma\alpha}{}^\mu T^\gamma{}_{\mu\beta} - \Gamma_{\gamma\beta}{}^\mu T^\gamma{}_{\alpha\mu} + \tfrac{1}{2}T^\gamma{}_{\mu\alpha}T^\mu{}_{\gamma\beta} + T^\gamma{}_\alpha{}^\mu T_{\gamma\beta\mu}\\ &- \tfrac{1}{2}T_\alpha T_\beta + (l^2/\kappa)F_{\alpha\nu\sigma\tau}F_\beta{}^{\nu\sigma\tau} - \tfrac{1}{4}\eta_{\alpha\beta}(T^{\gamma\sigma\mu}T_{\gamma\sigma\mu} - T^\gamma T_\gamma + (l^2/\kappa)F_{\mu\nu\sigma\tau}F^{\mu\nu\sigma\tau})\\ ={}& 0\end{aligned} \tag{2.17}$$

where κ and l^2 are coupling constants, and

$$\begin{aligned}\tau^\gamma{}_{\alpha\beta} \equiv{}& \partial_\mu F^{\gamma\mu}{}_{\alpha\beta} - \Gamma_{\nu\alpha}{}^\mu F^{\gamma\nu}{}_{\mu\beta} + \Gamma_{\nu\beta}{}^\mu F^{\gamma\nu}{}_{\mu\alpha} + \Gamma_\mu F^{\gamma\mu}{}_{\alpha\beta} + (\Gamma_{\nu\mu}{}^\gamma + \tfrac{1}{2}T_{\mu\nu}{}^\gamma)F^{\mu\nu}{}_{\alpha\beta} + (\kappa/l^2)T^\gamma{}_{[\alpha\beta]}\\ ={}& 0.\end{aligned} \tag{2.18}$$

The procedure in looking for solutions of (2.17) and (2.18) is to regard them as equations for the 'unknown' functions e_i^α and $F_{\alpha\beta\gamma}$. In analogy with the spherically symmetric solutions of Baekler *et al* (1980) and Baekler (1982) we restrict ourselves to solutions of (2.17) and (2.18) for which the metric components $g_{ij} = \eta_{\alpha\beta}e_i^\alpha e_j^\beta$ are solutions of Einstein's equations with the cosmological constant:

$$\tilde{R}_{ij} = \Lambda g_{ij} \tag{2.19}$$

where $\tilde{R}_{ij}$ is the Ricci tensor for the symmetric Riemannian connection defined by g_{ij} and $\Lambda = \pm 3\kappa/4l^2$ (see equation (7.5) of Baekler *et al* (1980) where, however, it is only the + sign that occurs). The first step is therefore to solve (2.19) for static, cylindrical and plane symmetric space–times.

3. Einstein equations with the cosmological constant

Consider a static space–time which, in addition to the timelike hypersurface-orthogonal Killing vector field, has two spacelike Killing fields. Furthermore we assume that the three Killing fields are mutually orthogonal and commute among themselves. One can then choose the coordinates so that

$$ds^2 = -\exp(2u)\,dt^2 + \exp(2v)\,dy^2 + \exp(2w)\,dz^2 + dx^2, \tag{3.1}$$

where u, v and w are functions of x only. If the coordinate lines of y (say) are closed with $0 \leqslant y \leqslant 2\pi$ and $-\infty < z < \infty$, $0 < x < \infty$, the metric is cylindrically symmetric with y as the angular, x the cylindrical radial and z the longitudinal coordinate. If $-\infty < x, y, z < \infty$, the symmetry may be called pseudo-planar (see Bronnikov and Kovalchuk 1979). From the point of view of the local field equations both cases may be treated simultaneously. For the vacuum field equations with zero cosmological constant one may transform to Weyl canonical coordinates with only two independent functions in the metric. However, this is not possible here.

The field equations (2.19) for the metric (3.1) yield

$$v'' + w'' + v'^2 + w'^2 + v'w' = \Lambda \tag{3.2}$$

$$w'' + u'' + w'^2 + u'^2 + u'w' = \Lambda \tag{3.3}$$

$$u''+v''+u'^2+v'^2+u'v'=\Lambda \tag{3.4}$$

$$v'w'+w'u'+u'v'=\Lambda \tag{3.5}$$

where a prime denotes differentiation with respect to x. Let $\xi=u+v+w$, $\eta=u-v$, $\beta=v-w$ and $\alpha=w-u$. Then (3.2)–(3.5) give

$$\xi''+\xi'^2=3\Lambda \tag{3.6}$$

$$\eta'=a\,\exp(-\xi) \qquad \beta'=b\,\exp(-\xi) \qquad \alpha'=c\,\exp(-\xi) \tag{3.7}$$

where a, b and c are constants of integration, with

$$a+b+c=0. \tag{3.8}$$

We distinguish the cases for which $\Lambda>0$ and $\Lambda<0$.

3.1. Case 1: $\Lambda>0$

The general solution of (3.6) is

$$\xi=\ln(g\,\exp(qx)+d\,\exp(-qx)) \tag{3.9}$$

where $q=(3\Lambda)^{1/2}$; g and d are constants. The functions η, β and α are then obtained from (3.7) by a simple quadrature and hence u, v and w are determined. On substituting (3.7) and (3.9) into (3.2)–(3.5) one obtains

$$a^2+b^2+c^2=-8gdq^2. \tag{3.10}$$

We therefore have two subcases: case 1(a) for which $d=0$, $g\neq 0$ (or $g=0$, $d\neq 0$) and consequently, by (3.10), $a=b=c=0$, and case 1(b) for which both g and d are non-zero and, by (3.10), necessarily of opposite sign.

By some manipulation and rescaling of the coordinates one finally obtains the following forms for the functions in the metric (3.1).

Case 1(a). ($d=0, g\neq 0$):

$$u=v=w=qx/3. \tag{3.11}$$

If $g=0$, $d\neq 0$, then $qx/3$ is replaced by $-qx/3$.

Case 1(b). ($g\neq 0, d\neq 0$):

$$e^u=(\sinh(qx)f(x)^A)^{1/3} \tag{3.12}$$

$$e^v=(\sinh(qx)f(x)^B)^{1/3} \tag{3.13}$$

$$e^w=(\sinh(qx)f(x)^C)^{1/3} \tag{3.14}$$

where

$$f(x)=(\cosh(qx)-1)/(\cosh(qx)+1) \tag{3.15}$$

and

$$A+B+C=0 \qquad A^2+B^2+C^2=\tfrac{3}{2}. \tag{3.16}$$

By (3.16) the constants A, B and C may be expressed in terms of a single parameter p as

$$A=\pm 3^{1/2}/[2(1+p+p^2)^{1/2}] \qquad B=pA \qquad C=-(1+p)A. \quad (3.17)$$

3.2. *Case 2: $\Lambda<0$*

The general solution of (3.6) is then

$$\xi=\ln(g \sin q(x+\varepsilon)) \quad (3.18)$$

where $q=(-3\Lambda)^{1/2}$, g and ε are constants. Again with some manipulation one can express the metric in the form (3.1) with

$$\mathrm{e}^u=(\sin(qx)f(x)^A)^{1/3} \quad (3.19)$$

$$\mathrm{e}^v=(\sin(qx)f(x)^B)^{1/3} \quad (3.20)$$

$$\mathrm{e}^w=(\sin(qx)f(x)^C)^{1/3} \quad (3.21)$$

where

$$f(x)=(1-\cos(qx))/(1+\cos(qx)) \quad (3.22)$$

and A, B and C satisfy (3.16).

Note that for cylindrical symmetry, where y is the angular coordinate, the topological implications of rescaling y should be considered in all of the above cases.

Stationary, cylindrically symmetric solutions to Einstein's equations with the cosmological constant have been treated by Krasinski (1975). It is easy to verify that case 1(a) above is equivalent to the metric (9.3) of his paper, while a rather involved coordinate transformation shows cases 1(b) and 2 to be equivalent to his Type B solutions. However, the functions occurring in the Type B metrics of Krasinski are considerably more complicated, containing, as they do, seven (constant) parameters instead of the two parameters q and p of the present paper. Cases 1(b) and 2 with t and x interchanged also correspond to the spatially homogeneous solution of Saunders (see Kramer *et al* 1980).

4. Solutions of the QPG field equations

In this section we present some special solutions of the QPG field equations (2.17) and (2.18). The metric is taken to be of the form (3.1) and the obvious orthonormal tetrad field

$$e_i^0\,\mathrm{d}x^i=\mathrm{e}^u\,\mathrm{d}t \qquad e_i^1\,\mathrm{d}x^i=\mathrm{e}^v\,\mathrm{d}y \qquad e_i^2\,\mathrm{d}x^i=\mathrm{e}^w\,\mathrm{d}z \qquad e_i^3\,\mathrm{d}x^i=\mathrm{d}x \quad (4.1)$$

is chosen where $(x^0, x^1, x^2, x^3)=(t, y, z, x)$. We look for solutions of the equations (2.17) and (2.18) for which the functions u, v and w have the forms given in each of the three cases described in § 3, and $\Lambda=3\kappa/4l^2$ in case 1 and $\Lambda=-3\kappa/4l^2$ in case 2. On making the substitutions (4.1) with the prescribed forms of u, v and w in each case, equations (2.17) and (2.18) become equations for the torsion components $F_{\alpha\beta\gamma}=-F_{\beta\alpha\gamma}$. In order to have manageable equations restrictions will also be imposed on the $F_{\alpha\beta\gamma}$ which will be specified when we come to deal with each case in turn.

When written out in full the expression for $\Sigma_{\alpha\beta}$ and $\tau^{\gamma}{}_{\alpha\beta}$ occurring in equations (2.17) and (2.18) are very long and unwieldy. All the calculations have been done on a computer using a REDUCE program and a certain degree of trial and error was involved. As it would be extremely tedious to reproduce the details of the calculations we shall simply describe the procedure used and state the results.

Solution for case 1(a). Let e_i^α be given by (4.1) with u, v and w as in (3.11). The only non-zero components of $\Omega_{\alpha\beta\gamma}$ (modulo the antisymmetry, $\Omega_{\alpha\beta\gamma}=-\Omega_{\beta\alpha\gamma}$) are then $\Omega_{030}=-\Omega_{131}=-\Omega_{232}=q/3$. Using this as a guide we restrict ourselves to seeking solutions of (2.17) and (2.18) for which

$$\begin{aligned} &F_{030}=-F_{300}=U(x)\\ &F_{131}=-F_{311}=F_{232}=-F_{322}=-U(x) \end{aligned} \tag{4.2}$$

and $q^2=9\kappa/4l^2$.

Substitution of (4.2) and (4.1), with u, v and w as in (3.11), into (2.18) yields just one independent equation for $U(x)$:

$$U''+qU'=2U^2(U-q). \tag{4.3}$$

On substituting into (2.17) one obtains two independent equations:

$$3U'^2+2q(U+q)U'-U(3U^3-4qU^2+3q^2U-2q^3)=0 \tag{4.4}$$

and

$$9U'^2+2q(3U-q)U'-U(9U^3-12qU^2+q^2U+2q^3)=0. \tag{4.5}$$

Eliminate U'^2 from (4.4) and (4.5) to get

$$U'=U(U-q) \tag{4.6}$$

which has

$$U=q/(1-D\exp(qx)) \tag{4.7}$$

as its general solution, where D is an arbitrary constant. Finally one may verify that (remarkably!) (4.7) satisfies *all* of the equations (4.3)–(4.5).

Thus (4.1) and (4.2), with u, v, w and U given by (3.11) and (4.7), is a solution of the QPG equations (2.17) and (2.18).

Solution for case 1(b). Let e_i^α be as in (4.1) with e^u, e^v and e^w given by (3.12)–(3.15). The only independent non-zero components of $\Omega_{\alpha\beta\gamma}$ are again Ω_{030}, Ω_{131} and Ω_{232}. For this case we have so far looked only for solutions in which one of the independent components F_{030}, F_{131} or F_{232} is non-zero while all the other independent components vanish.

First of all let

$$\begin{aligned} &F_{030}=-F_{300}=U(x)\\ &\text{All other components of } F_{\alpha\beta\gamma}=0 \end{aligned} \tag{4.8}$$

and, as before, $q^2=9\kappa/4l^2$. It is found that the only independent non-zero components of $\tau^\gamma{}_{\alpha\beta}$ (equation (2.18)) are $\tau^0{}_{03}$, $\tau^1{}_{13}$ and $\tau^2{}_{23}$. The equation $\tau^0{}_{03}=0$ yields

$$\sinh^2(qx)U''+q\sinh(qx)\cosh(qx)U'-q^2U=0, \tag{4.9}$$

the general solution of which is

$$U=(a\cosh(qx)+b)/\sinh(qx) \tag{4.10}$$

where a and b are constants of integration. The solution $U(x)$ of (4.10) satisfies $\tau^1{}_{13}=0$ and $\tau^2{}_{23}=0$ if and only if $a=0$ and either

$$b=2q/3 \qquad A=1 \qquad B=-\tfrac{1}{2} \qquad C=-\tfrac{1}{2}$$

or

$$b=-2q/3 \qquad A=-1 \qquad B=\tfrac{1}{2} \qquad C=\tfrac{1}{2}.$$

Furthermore, with either of these two sets of values for the constants the equations $\Sigma_{\alpha\beta}=0$ (equation (2.17)) are satisfied automatically.

Thus (4.1) and (4.8), with u, v and w given by (3.12)–(3.15), is a solution of the QPG equations (2.17) and (2.18) if and only if either

$$U(x)=2q/3\sinh(qx) \qquad A=1, \quad B=-\tfrac{1}{2}, \quad C=-\tfrac{1}{2} \tag{4.11}$$

or

$$U(x)=-2q/3\sinh(qx) \qquad A=-1, \quad B=\tfrac{1}{2}, \quad C=\tfrac{1}{2}. \tag{4.12}$$

By a similar procedure the following solutions for the metric of case 1(b) may also be found:

$$F_{131}=-F_{311}=2q/3\sinh(qx) \qquad A=\tfrac{1}{2}, \quad B=-1, \quad C=\tfrac{1}{2} \tag{4.13}$$

or

$$F_{131}=-F_{311}=-2q/3\sinh(qx) \qquad A=-\tfrac{1}{2}, \quad B=1, \quad C=-\tfrac{1}{2}, \tag{4.14}$$

all other $F_{\alpha\beta\gamma}$ being equal to zero, and

$$F_{232}=-F_{322}=2q/3\sinh(qx) \qquad A=\tfrac{1}{2}, \quad B=\tfrac{1}{2}, \quad C=-1 \tag{4.15}$$

or

$$F_{232}=-F_{322}=-2q/3\sinh(qx) \qquad A=-\tfrac{1}{2}, \quad B=-\tfrac{1}{2}, \quad C=1, \tag{4.16}$$

all other $F_{\alpha\beta\gamma}$ being equal to zero.

Solution for case 2. An attempt to find solutions in this case along the lines of the preceding example proves to be unsuccessful. Take e_i^α as in (4.1) with e^u, e^v and e^w given by (3.19)–(3.22) and the torsion as in (4.8). Proceeding exactly as before, it is found that (2.18) is satisfied if and only if either

$$U(x)=2q(2\cos qx+3)/3\sin qx \qquad A=1, \quad B=-\tfrac{1}{2}, \quad C=-\tfrac{1}{2} \tag{4.17}$$

or

$$U(x)=2q(2\cos qx-3)/3\sin qx \qquad A=-1, \quad B=\tfrac{1}{2}, \quad C=\tfrac{1}{2}. \tag{4.18}$$

However, on substituting these solutions into (2.17) one obtains

$$\Sigma_{\alpha\beta}=-(2q^2/3)\eta_{\alpha\beta} \tag{4.19}$$

so there is no vacuum solution *of this form* for case 2. Similarly, further solutions of equation (2.18) are given by

$$F_{131}=-2q(2\cos qx-3)/3\sin qx \qquad A=\tfrac{1}{2}, \quad B=-1, \quad C=\tfrac{1}{2} \tag{4.20}$$

or

$$F_{131}=-2q(2\cos qx+3)/3\sin qx \qquad A=-\tfrac{1}{2}, \quad B=1, \quad C=-\tfrac{1}{2}, \tag{4.21}$$

all other independent $F_{\alpha\beta\gamma}$ being equal to zero, and the obvious corresponding solution for the case in which all the $F_{\alpha\beta\gamma}$ vanish except for $F_{232} = -F_{322}$. Substitution of these solutions into (2.17) again yields (4.19).

The lack of success in finding a vacuum solution for this case, where $\Lambda = -3\kappa/4l^2$ instead of $+3\kappa/4l^2$ as in case 1, would seem to indicate that, in general, the QPG equations are sensitive to the sign of Λ in (2.19).

Acknowledgment

I am grateful to Professor F W Hehl for introducing me to the QPG theory and for several useful suggestions, also to Dr P Baekler who with Professor Hehl independently checked the solutions of § 3 and solution for case 1(a) in § 4.

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